The Greither unit index, an undergraduate overview and a *sagemath* implementation

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Abstract

In this work I show a rielaboration of the article:

Cornelius Greither, *Improving Ramachandra's and Levesque's unit index*, 1999

I've introduced the principal instruments and results from Galois Theory and Number Theory necessary for the comprehension of the article, in particular the circular units, the class number and the Dirchlet charactes. The aim of the article is to generalize the results of other mathematichans in the creation of a subgroup of the unit group E_K with the same rank, in particular it uses a general function β in the group ring $\mathbb{Z}[G_0]$ to get different groups. Then we study a particular choice of β, that define an optimal group with a simple index $[E_K, C_\beta]$.
After this, in a *jupyter* file, I've implemented some of the calculations.

Contents

1 Introduction to the working set

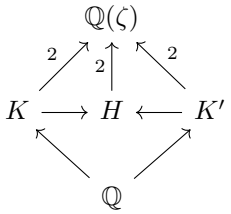
Consider the n-th cyclotomic field $\mathbb{Q}(\zeta_n)$ with ζ_n a n-th primitive root of unity, with $n \not\equiv 2 \mod 4$, and define K as the maximal real subfield of $\mathbb{Q}(\zeta)$, also another notation that we will use for the maximal real subfield is $\mathbb{Q}(\zeta_n)^+$. From now we will refer to ζ_n without the index if not necessary.

Proposition 1.1. *The maximal real subfield is* $K = \mathbb{Q}(\zeta + \zeta^{-1})$

Proof. First of all we can easly see that K is real, infact since for the root of unity $\overline{\zeta} = \zeta^{-1}$ (complex conjugation) and so:

$$\overline{\zeta + \zeta^{-1}} = \overline{\zeta} + \overline{\zeta^{-1}} = \zeta^{-1} + \zeta$$

So $\zeta + \zeta^{-1}$ is real and K too.
Since $\mathbb{Q}(\zeta)$ is not real we have that the index $e := [\mathbb{Q}(\zeta) : K] \geq 2$ (strictly greater than 1).
Consider now the polynomial of degree 2 in $K[x]$: $f = (x - \zeta)(x - \zeta^{-1}) = x^2 - (\zeta + \zeta^{-1})x + 1$, since ζ is a root obviosly $e \leq 2$, so the subfield K has maximal degre since this is the minimal degree for a proper subfield.
If there was another $K' = \mathbb{Q}(\chi)$ with such property we can consider $H = \mathbb{Q}(\zeta, \chi)$ that is also real with $\mathbb{Q}(\zeta) \supsetneq H \supset K$, so $H = K$ and akin $H = K'$ so $K = K'$ and K is unique. □



Now we will consider the **group of units** E_K that is the group formed by the invertible elements of its ring of integers O_K^*. Is it possible to characterize the ring of integers for K [6, Proposition 2.16] similiarly to what happens for $O_{\mathbb{Q}(\zeta)}$ (infact the proof follows without difficulty from this)

Proposition 1.2. $O_K = \mathbb{Z}[\zeta + \zeta^{-1}]$

Since $x^n - 1$ is separable $\mathbb{Q}(\zeta)/\mathbb{Q}$ is a Galois extension and it's easy to see that its Galois group G_0 is isomorphic to $(\mathbb{Z}_n)^*$. Also we can see that:

Proposition 1.3. *$K/\mathbb{Q}$ is a Galois extension and its Galois group G is isomorphic to $\mathbb{Z}_n^*/\{\pm 1\}$*

Proof. Consider the map $\sigma : G_0 \to G$ that maps α_i to $\alpha_{i|_K}$ where α_i is the automorphism that maps ζ to ζ^i. Obviously σ is a morphism of groups. Also

it is easy to describe its kernerl:

$$\begin{aligned}\ker(\sigma) =&\{\alpha_i \in G_0 \,|\, x = \alpha_i(x) \text{ for all } x \in K\}\\ &\overset{(1)}{=}\{\alpha_i \in G_0 \,|\, \zeta + \zeta^{-1} = \alpha_i(\zeta + \zeta^{-1}) = \zeta^i + \zeta^{-i}\}\\ &\overset{(2)}{=}\{\alpha_1, \alpha_{-1}\}\end{aligned}$$

Where (1) follows from the fact that $K = \mathbb{Q}(\zeta+\zeta^{-1})$ and (2) from linear algebra. So from the first theorem of isomorphism $\sigma(G_0) \simeq \mathbb{Z}_n^*/\{\pm 1\}$ and then

$$\phi(n)/2 = |\mathbb{Z}_n^*/\{\pm 1\}| \leq |G| \leq [K : \mathbb{Q}] = [\mathbb{Q}(\zeta) : \mathbb{Q}]/2 = \phi(n)/2$$

So $\sigma(G_0) = G$ and $|G| = [K : \mathbb{Q}]$ and the thesis follows. □

Remark 1. We excluded the case of $n \equiv 2 \mod 4$ because it is a repetition, infact in this situation $G_0 \simeq \mathbb{Z}_{2+4k}^*$ and since $2+4k = 2(1+2k)$ with the second term odd for the Chinese reminder theorem $\mathbb{Z}_{2+4k}^* \simeq \mathbb{Z}_2^* \times \mathbb{Z}_{1+2k}^* \simeq \{1\} \times \mathbb{Z}_{1+2k}^* \simeq \mathbb{Z}_{1+2k}^*$ that is isomorphic to the Galois group for the $n/2$-th root of unity.

1.1 The circluar units and the class number

Definiton 1.4. If $\mathbb{K}$ is a number field (as $\mathbb{Q}(\zeta)$ and K) we can define the **ideal class group** as the quotient $\mathcal{F}_{\mathbb{K}}/\mathcal{P}_{\mathbb{K}}$ where:

- $\mathcal{F}_{\mathbb{K}}$ is the group of the nonzero fractional ideals of the ring of integers $O_{\mathbb{K}}$, that are the $O_{\mathbb{K}}$-submodules J of $\mathbb{K}$ such that exists $r \in O_{\mathbb{K}}$ such that $rI \subset O_{\mathbb{K}}$
- $\mathcal{P}_{\mathbb{K}}$ is the set of nonzero principal fractionary ideals, so the ideals generated by only one element

We will indicate the number of classes in $\mathcal{F}_{\mathbb{K}}/\mathcal{P}_{\mathbb{K}}$ as h_K. This number will measure the *"distance"* of $O_{\mathbb{K}}$ to became a unique factorization domain. In [4, Page 141] it is proven that actually the ideal class group is finite so h_K is well defined.

Definiton 1.5. For a field $\mathbb{K} \subseteq \mathbb{Q}(\zeta_n)$ (with n minimal) we define the group of cyclotomic (or circular) units as the intesection $C_{\mathbb{K}}$ of the group generated by:

$$\{-1, \zeta, 1 - \zeta^a \text{ for } a = 1, ..., n-1\}$$

and the unit group of $O_{\mathbb{K}}$ ($E_{\mathbb{K}}$). An elements of $C_{\mathbb{K}}$ is said to be a **circular unit** of $\mathbb{K}$.

In general the circular units aren't easy to describe, infact in general $1 - \zeta^a$ is not a unit, but for the particular case in which $\mathbb{K}$ is the maximal real subfield (K) it has some intresting properties and it's related to the class number.

If $n = p^m$ where p is a prime it is possible to describe ([6, Lemma 8.1, Theorem 8.2]) explicitly the group of circluar units as the group generated by -1 and:

$$\xi_a = \zeta^{\frac{1-a}{2}} \frac{1-\zeta^a}{1-\zeta} \text{ for } 1 < a < \frac{p^m}{2}, (a, p) = 1$$

Also we have the equality for the index:

$$[E_K : C_K] = h_K$$

Moreover Sinnot in [5] has generalized this to arbitrary n by showing that E_K/C_K is finite and the index is:

$$[E_K : C_K] = 2^a h_K$$

where if g is the number of distinct primes dividing n we have that $a = 0$ if $g = 1$ (as expected) and $a = 2^{g-2} + 1 - g$ otherwise. Even if the index is simple does not exist a simple costruction of C_K, so we have the problem:

Explicitly construct a group C' with finite index $[E_K : C']$ that is *optimal*

Where we will understand later what we mean by *optimal*, but essentially we want the index to be small and with a simple factorization for $[E_K : C']/h_K$. In particular the costruction of Greither will generalize the work of Ramachandra and Levesque, so we will omit them from now and see them later.

1.2 Dirichlet Characters

Definiton 1.6. Given a group X and a field $\mathbb{F}$ a Dirichlet character is a group homomorphism $\chi : X \to \mathbb{F}^*$

In our case the field is $\mathbb{C}$ and X is the Galois group $G_0 \simeq \mathbb{Z}_n^*$, so we can see the Dirichlet characters as homomorphisms: $\xi : \mathbb{Z}_n^* \to \mathbb{C}^*$. Since if $n|m$ there is a natural homomorphism $\mathbb{Z}_m^* \to \mathbb{Z}_n^*$, composing it with χ we can induct a new character $\hat{\chi} : \mathbb{Z}_m^* \to \mathbb{C}^*$. This characters are completely equivalent. So we can define k to be minimal positive integer such that exists a character $\chi' : \mathbb{Z}_k^* \to \mathbb{C}^*$ equivalent to χ, and call it the **conductor** of χ, denoted by $f_\chi := k$.

In some cases the character are also extended as ring homomorphisms from $\mathbb{Z}_n \to \mathbb{C}$, assuming χ to be zero on the non invertible elements. In this way the conductor can be seen as a sort of period, infact for all n we have $\chi(n) = \chi(n + f_\chi)$.

Also we need another object: the group ring $\mathbb{Z}[G]$, that is a free $\mathbb{Z}$-module with G as basis on which we define the addition (using the module addition) and the moltiplication inducting it from the operation of G. This costruction is also possible for a general ring and a multiplicative group:

Definiton 1.7. The group ring of X over R, denoted by $R[X]$ or RX, is the set of all mapping $f : X \to R$ with finite support (i.e. with finite $x \in X$ such that $f(x) \neq 0$). The addition and the scalar multiplication are defined as usual.

We can also have a group structure over $R[X]$ using the vector addition and the multiplication: were fg is defined as: $fg(x) = \sum_{y \in X} f(y)g(y^{-1}x) = \sum_{uv=x} f(u)g(v)$.

This is only a formal representation of the linear combinations, useful for the definition, but we will obviosly use a simpler notation $f = \sum_{x \in X} f(x)x$.

Now we would like to generalize again the characters as ring homorphism from $\mathbb{Z}[G]$ (or another Galois group) to $\mathbb{C}$. This is very simple since G is a basis for the free $\mathbb{Z}$-module its definition over the group is enougth.

Notation. Given the elements $z \in \mathbb{Q}(\zeta)$ and $f \in \mathbb{Z}[G_0]$ it's well defined the power notation z^f, infact for $g \in G_0$ we have the well definiton for $z^g = g(z)$, $z^{g_1+g_2} = z^{g_1} z^{g_2}$ and $z^{-g} = (z^g)^{-1}$.

2 The Greither Setup

Let's consider an integer n (with $n \not\equiv 2 \mod 4$), with factorization $n = p_1^{e_1} \cdots p_s^{e_s}$ and let $S = \{1, ..., s\}$. We will use the power set $\mathcal{P}_S = \{I \,|\, I \subsetneq S\}$ and the notation $n_I = \prod_{i \in I} p_i^{e_i}$

The Greither's idea is to define a subgroup starting from a function $\beta : \mathcal{P}_S \to \mathbb{Z}[G_0]$, then varing β we have different subgroups but with similiar properties.

Definiton 2.1. A function β is called multiplicative if $\beta(\emptyset) = 1$ and for all sets I, J with empty intersection we have $\beta(I \cup J) = \beta(I)\beta(J)$.

A multiplicative function is uniquely determinated by its values on the singletons: $\{\{i\}\,|, i \in S\}$ (we will use this later for a particular construction)

Consider a general function β and $I \in \mathcal{P}_S$, we define $z_I := 1 - \zeta^{n_I}$ and

$$z(\beta) := \prod_{I \in \mathcal{P}_S} z_I^{\beta(I)}$$

Using that $1 - \zeta^{-m} = -\zeta^{-m}(1 - \zeta^m)$, $\overline{\zeta} = \zeta^{-1}$ and the properties of complex conjugation we have that

$$\overline{z(\beta)} = \prod_{I \in \mathcal{P}_S} (1 - \zeta^{-n_I})^{\beta(I)} = \prod_{I \in \mathcal{P}_S} -\zeta^{-n_I\beta(I)}(1 - \zeta^{n_I})^{\beta(I)} =$$
$$= (-1)^{|\mathcal{P}_S|} \prod_{I \in \mathcal{P}_S} \zeta^{-n_I\beta(I)} z_I^{\beta(I)} \overset{*}{=} -\zeta^{-t} z(\beta) \text{ with } t = \sum n_I \beta(I) \quad (1)$$

In $*$ we use that $|\mathcal{P}_S| = 2^s - 1$ is odd.

We define now for $a \in (1, n/2)$ coprime with n the real unit:

$$\xi_a(\beta) := \zeta^{d_a(\beta)} \frac{\sigma_a(z(\beta))}{z(\beta)} \text{ with } d_a(\beta) = (1 - a)\frac{t}{2} \quad (2)$$

Where σ_a is the automorphism $\zeta \mapsto \zeta^a$. This is real because using the equation 1 and $\overline{\sigma_a(z)} = \sigma_a(\overline{z})$ we have:

$$\overline{\xi_a(\beta)} = \zeta^{-d_a(\beta)} \frac{\zeta^{-at}\sigma_a(z(\beta))}{\zeta^{-t} z(\beta)} = \xi_a(\beta) \quad (3)$$

And its a unit because its the product of circular units. We now use this units to define the goal group of the article:

C_β is the group generated by *-1* and $\xi_a(\beta)$ for $1 < a < n/2$ and $(a, n) = 1$

For its index we will use the notation: $[E_K : C_\beta] = h_K i_\beta$.

2.1 A little remark

Sometimes it is easier to work with functions β to $\mathbb{Z}[G]$ instead of $\mathbb{Z}[G_0]$ and than consider a lift $\overline{\beta} : \mathcal{P}_S \to \mathbb{Z}[G_0]$. Obviously this is not unique, but this is not a problem because we can show that C_β remain the same.

Remark 2. Given two set Z, Y and a function $\phi : Y \to Z$ we say that $f' : X \to Y$ is a **lift** of $f : X \to Z$ if $f = \phi \circ f'$, i.e. if the following diagram commute:

$$\begin{array}{ccc} & & Y \\ & \overset{f'}{\nearrow} & \downarrow \phi \\ X & \xrightarrow[f]{} & Z \end{array}$$

In our case ϕ is the morphism inducted on the group ring by the projection $G_0 \simeq \mathbb{Z}_n^* \to \mathbb{Z}_n^* / \pm 1 \simeq G$

Initally we can osserve that we can factor the real unit $\xi_a(\beta)$ with simpler real units

$$x_a(\beta, I) = \zeta^{\frac{(1-a)}{2} n_I \beta(I)} \frac{\sigma_a(z_I^{\beta(I)})}{z_I^{\beta(I)}}$$

such that we have the equality:

$$\xi_a(\beta) = \prod_{I \in \mathcal{P}_S} x_a(\beta, I) \tag{4}$$

Lemma 2.2. *Consider two functions β_1 and β_2 from $\mathcal{P}_S$ to $\mathbb{Z}[G_0]$ such that for all $I \in \mathcal{P}_S$ their images of $\beta_i(I)$ coincides in $\mathbb{Z}[\mathrm{Gal}(\mathbb{Q}(\zeta_{n/n_I})^+/\mathbb{Q})]$* [1] *for $i = 1, 2$. Then for all $I \in \mathcal{P}_S$ $x_a(\beta_i, I)$ coincides for $i = 1, 2$*

Proof. Obviously for all $I \in \mathcal{P}_S$ $x_a(\beta_i, I)$ depends only on the image of β_i over $z_I = 1 - \zeta_n^{n_I} \in \mathbb{Q}(\zeta_{n/n_I})$ [2], so it's enougth to show the equivalence over $\mathbb{Q}(\zeta_{n/n_I})$. Since the two functions are equal on $\mathbb{Q}(\zeta_{n/n_I})^+$ their difference $\beta_1(I) - \beta_2(I)$ is the identity on the reals, so it is a multimple of $1 - j$, where j is the complex conjugation We can observe now, using morphism properties, that exist a unit r such that:

$$\mathbb{Q}(\zeta_{n/n_I})^+ \ni q = \frac{x_a(\beta_1, I)}{x_a(\beta_2, I)} = \left(\zeta^{\frac{(1-a)}{2} n_I} \frac{\sigma_a(z_I)}{z_I} \right)^{\beta_1(I) - \beta_2(I)} = r^{1-j}$$

[1] Observe that $\mathbb{Q}(\zeta_{n/n_I})^+$ is a subfield of K since $\zeta_{n/n_I} = \zeta_n^{n_I}$, and since we see the elements of the group rings as homomorphism of fields make sense to compare two elements for their image on $\mathbb{Q}(\zeta_{n/n_I})^+$

[2] $\zeta_{n/n_I} = \zeta_n^{n_I}$

So we have that $\overline{q} = q^j = r^{(1-j)j} = r^{j-1} = q^{-1}$ (since $j^2 = 1$), that for real numebers happen only for ± 1 □

Remark 3. For what we have seen in the equation 4 it follows immediatly that also $\xi_a(\beta)$ is unique up to a sign if β is a lifting of a function from $\mathcal{P}_S$ to $\mathbb{Z}[G]$. Since the group C_β contains -1 it is enougth to have a function $\beta : \mathcal{P}_S \to \mathbb{Z}[G]$ for its definition.

3 Index calculation

Theorem 3.1. *For any function $\beta : \mathcal{P}_S \to \mathbb{Z}[G]$ we have*

$$i_\beta = \prod_{\substack{\chi \neq 1 \\ even}} \left(\sum_{\substack{I \in \mathcal{P}_S \\ (f_\chi, n_I)=1}} \phi(n_I) \cdot \chi(\beta(I)) \cdot \prod_{i \notin I} (1 - \chi^{-1}(p_i)) \right) \tag{5}$$

Remarks 4 (On theorem 3.1).

- ϕ is the Euler totient function
- A character χ is said to be **even** if $\chi(-1) = 1$
- With χ^{-1} we mean the character defined as $1/\chi$ on the invertible elements and zero otherwhise, that is also a morphism because $1/(xy) = (1/x)(1/y)$.

For the proof we need the following Lemmas:

Lemma 3.2. *For $z \in \mathbb{Q}(\zeta)^*$ and $\gamma \in \mathbb{Z}[G_0]$, then for any character χ we have:*

$$\sum_{(a,n)=1} \chi^{-1}(a) \log |z^{\sigma_a \gamma}| = \chi(\gamma) \sum_{(a,n)=1} \chi^{-1}(a) \log |z^{\sigma_a}| \tag{6}$$

Proof. It is easy to prove this for $\gamma = \sigma_g \in G_0$, infact since g is invertible in $\mathbb{Z}_n$ is possible to change the index from $(a, n) = 1$ to $(ag, n) = 1$ and rearrange. Then we can pass to $\mathbb{Z}[G_0]$ using the additivity of χ and the logaritm of exponential (also the modulo is multiplicative). □

For the calculation of the index we need a new object that allows to evaluate a :

Definiton 3.3. The **regulator** R_L of a number fields L is defined as follows: given its rank r, a set of independent units $\{\epsilon_1, ..., \epsilon_r\} \subset L$ and $\{\sigma_1, ..., \sigma_{r+1}\}$ its embedding into $\mathbb{R}$ or $\mathbb{C}$. Set δ_j to be 1 if σ_j is real, and 2 otherwhise.
Then:

$$R_L(\epsilon_1, ..., \epsilon_r) = |\det(\delta_i \log |\epsilon_j^{\sigma_i}|)_{1 \leq i,j \leq r}| \tag{7}$$

Remark 5. The embedding that we decide to omit is not relevant, infact since they are units their norm is 1, so $\sum_i \delta_i \log |\epsilon_j^{\sigma_i}| = \log |\prod_i \epsilon_j^{\delta_i \sigma_i}| = \log |N(\epsilon_j)| = 0$, so writing this equality as a linear system from Cramer formula follows the uniqueness of the determinant up to a sign.

Now we need to recall some Lemmas from [6] without the proofs:

Lemma 3.4 (Lemma 4.15 in [6]). *Given the groups $A \subset B$ of finite index, generated by independent units of a number field L, respectively $\{\epsilon_i\}_{i=1}^r$ and $\{\mu_i\}_{i=1}^r$:*

$$[B : A] = \frac{R_L(\epsilon_1, ..., \epsilon_r)}{R_L(\mu_1, ..., \mu_r)} \tag{8}$$

Lemma 3.5 (Lemma 5.26 in [6]). *Let X be a finite abelian group and let f be a function on X with values in $\mathbb{C}$*

$$\det(f(\sigma\tau^{-1}) - f(\sigma))_{\sigma,\tau\neq 1} = \prod_{\substack{\chi\in\hat{X}\\ \chi\neq 1}} \sum_{\sigma\in X} \chi(\sigma) f(\sigma) \tag{9}$$

Where $\hat{X}$ is the set of homorphisms (characters) from X to $\mathbb{C}^*$

In our case X will be $G \equiv \mathbb{Z}_n/\pm 1$, and so the elements of $\hat{X}$ are the even characters of $\mathbb{Z}_n$.

Proof of Theorem 3.1 . Using Lemma 3.4 we can evaluate $[E_K : C_\beta]$ with the quotient of the regulators. In the equation 8 we can omit the unit -1 since it is contained in both the two groups (for what we have said in 5 can only change a sign).

So we need to prove that $R(\xi_a(\beta)) = \pm R_K h_K A$ with $(a, n) = 1\,,\ 1 < a < n/2$ and A be the right part of the equation 5. The $\pm$ is a more simple way to indicate that we don't matter the sign without inserting everything in a modulo.

From definition, using that δ_i is always 1 since the units are all real and the embeddings can be seen as elements of the Galois Group G:

$$\begin{aligned}
R(\xi_a(\beta)) &= \pm \det[\log|\xi_a(\beta)^\tau|] \quad ((a,n) = 1\,,\ 1 < a < n/2\,;\ \tau \in G)\\
&\overset{(1)}{=} \pm \det[f(\tau\sigma) - f(\tau)]_{\sigma,\tau\in G-1} \quad \text{with } f(\sigma) = \log|\sigma z(\beta)|\\
&= \prod_{\substack{\chi\neq 1\\ \text{even}}} \frac{1}{2} \sum_{(a,n)=1} \chi^{-1}(a) \log|\sigma_a z(\beta)| \qquad \text{using Lemma 3.5}\\
&= \prod_{\substack{\chi\neq 1\\ \text{even}}} \frac{1}{2} \sum_{(a,n)=1} \chi^{-1}(a) \sum_{I\in\mathcal{P}_S} \log|(1-\zeta^{n_i a})^{\beta(I)}|\\
&= \prod_{\substack{\chi\neq 1\\ \text{even}}} \frac{1}{2} \sum_{I\in\mathcal{P}_S} \left(\sum_{(a,n)=1} \chi^{-1}(a) \log|(1-\zeta^{n_i a})^{\beta(I)}| \right)\\
&\overset{6}{=} \prod_{\substack{\chi\neq 1\\ \text{even}}} \frac{1}{2} \sum_{I\in\mathcal{P}_S} \left(\chi(\beta(I)) \sum_{(a,n)=1} \chi^{-1}(a) \log|(1-\zeta^{n_i a})| \right)
\end{aligned}$$

Where in (1) we have used that $\log|\zeta^d| = 0$ because ζ is a unit and the logaritm's properties.

The last part is a bit technical and uses [6, Lemma 8.4] to reduce the first sum to the $I \in \mathcal{P}_S$ such that $(f_\chi, n_I) = 1$, and then continues as for the proof of Theorem 8.3 in [6, Pages 148-150] and involves the analytic class numebr formula and Dirichlet L-series (also Chapter 4 in [6]). □

3.1 Particular case of formula 5

Now we can try to see what happen if we request some conditions over β, with some particular cases.

Theorem 3.6. *If we assume $\beta : \mathcal{P}_S \to \mathbb{Z}[G]$ to be multiplicative then:*

$$i_\beta = \prod_{\substack{\chi \neq 1 \\ even}} \left(\prod_{p_i \nmid f_\chi} \left(\phi(p_i^{e_i}) \cdot \chi(\beta(i)) + 1 - \chi^{-1}(p_i) \right) \right) \tag{10}$$

Where $\beta(i)$ mean $\beta(\{i\})$

Proof. It is easy that we can lift β to $\mathbb{Z}[G_0]$ conserving multiplicativity. Consider now, for $\chi \neq 1$ even, the two factors :

$$T_\chi = \sum_{\substack{I \in \mathcal{P}_S \\ (f_\chi, n_I) = 1}} \phi(n_I) \cdot \chi(\beta(I)) \cdot \prod_{i \notin I} (1 - \chi^{-1}(p_i)) \tag{11}$$

and

$$U_\chi = \prod_{p_i \nmid f_\chi} \left(\phi(p_i^{e_i}) \cdot \chi(\beta(i)) + 1 - \chi^{-1}(p_i) \right)) \tag{12}$$

that are the arguments of the products in equations 5 and 10. So it's enough to prove $U_\chi = T_\chi$. Initially we can observe that the argument of the sum in 11 are the subset of $S_\chi = \{ i \,|\, p_i \nmid f_\chi \}$. Also we can observe

$$\phi(n_I) = \prod_{i \in I} \phi(p_i^{e_i})$$

$$\chi(\beta(I)) = \chi \left(\prod_{i \in I} \beta(i) \right) = \prod_{i \in I} \chi(\beta(i))$$

From which expanding the product of U_χ we get the equality. □

Using this formula and the definition of C_β we can see that for $\beta \equiv 1$ (that is the simplest example of multiplicative β) we get the Ramachandra's unit index from [3] (or in a more modern notation [6, Theorem 8.3]):

$$[E_K : C_R] = h_K \cdot \prod_{\substack{\chi \neq 1 \\ \text{even}}} \left(\prod_{p_i \nmid f_\chi} \left(\phi(p_i^{e_i}) + 1 - \chi(p_i) \right) \right) \tag{13}$$

Where C_R is the group generated by -1 and the units of the form of 2 with $\beta(I) = 1$:

$$\xi_a := \zeta^{d_a} \prod_{I \in \mathcal{P}_S} \frac{1-\zeta^{an_I}}{1-\zeta^{n_I}} \text{ with } d_a = \frac{1}{2}(1-a) \sum_{I \in \mathcal{P}_S} n_I$$

We can also construct β multiplicative such that:

$$\beta(i) = \begin{cases} 1 \text{ if exists } \chi \neq 1 \text{ even, with } \chi(p_i) = 1 \\ 0 \text{ otherwhise} \end{cases}$$

And we obtain the Levesque group $C_{\mathcal{D}}$ defined in [2, Page 331]

3.2 A new system of units

Following the previous steps we know construct a new multiplicative map β with a more optimal index.

Notation. If x is an element of finite group Γ we define:

$$N_x := 1 + x + ... + x^{\text{ord}(x)-1} \in \mathbb{Z}[\Gamma]$$

This will be called *trace* element of x.

Let now define G_i for $i = 1, ..., s$ to be the Galois group $\text{Gal}(\mathbb{Q}(\zeta_{n/p_i^{e_i}})^+/\mathbb{Q}))$. Consider now the Frobenius automorphism $F_i \in G_i$:

$$\begin{aligned} F_i : \mathbb{Q}(\zeta_{n/p_i^{e_i}})^+ &\longrightarrow \mathbb{Q}(\zeta_{n/p_i^{e_i}})^+ \\ \zeta_{n/p_i^{e_i}} &\longmapsto \zeta_{n/p_i^{e_i}}^{p_i} \end{aligned}$$

and its trace element $N_{F_i} \in \mathbb{Z}[G_i]$. Now we choose for every $i = 1, ..., s$ a lift of N_{F_i} into $\mathbb{Z}[G_0]$[3] and associate it to $\beta(i)$; then β is defined multiplicatively.

Of course β is not unique, but for all $I \in \mathcal{P}_S$ they coincide in $\mathbb{Z}[\text{Gal}(\mathbb{Q}(\zeta_{n/n_I})^+/\mathbb{Q})]$, so we can use Lemma 2.2 and C_β is well defined.

3.3 A factorization for i_β

Here we will recall some facts and definitions from [4, Chapter 11]. These are genaralities for a finite separable extension of a number field, but we will restrict in the case of $\mathbb{K}/\mathbb{Q}$ number field.

Consider a prime p in $\mathbb{Z}$ and its ideal extension in the ring of integers $pO_{\mathbb{K}}$. Since O_K is a Dedekin domain we can factorize it with prime ideals:

$$pO_{\mathbb{K}} = \prod_{j=1}^{g} \mathfrak{p}_j^{\epsilon_j} \tag{14}$$

[3] Remind that $\zeta_{n/p_i^{e_i}} = \zeta_n^{p_i^{e_i}}$

Definiton 3.7. The number g is said to be the *decomposition degree* (or number) of p in the extension $\mathbb{K}/\mathbb{Q}$.

For every $j = 1, ..., g$, ϵ_j is said to be the *ramification degree* (or index) of $\mathfrak{p}_j$ in $\mathbb{K}/\mathbb{Q}$.

For every $j = 1, ..., g$, $f_j := [O_{\mathbb{K}}/\mathfrak{p}_j : \mathbb{Z}_p]$ is called the *inertial degree* (or residual).

In particular is possible to prove that if $n = [\mathbb{K} : \mathbb{Q}]$ so

$$n = \sum_{j=1}^{g} f_j \epsilon_j$$

Also if $\mathbb{K}/\mathbb{Q}$ is a Galois extension then also ϵ_j, f_j does not depend on j and so

$$n = \epsilon f g$$

We also recall from [6, Theorem 3.7] the relation between the characters X over the galois group of $\mathbb{K}/\mathbb{Q}$ and decomposition degree of p in the extension $\mathbb{K}/\mathbb{Q}$:

$$g = |\{\chi \in X \mid \chi(p) = 1\} \tag{15}$$

We have now the ingredients for evaluating in a optimal way i_β.

For $i \in 1, ..., s$ define g_i, f_i, ϵ_i to be as in the definition 3.7 for the prime p_i in $K/\mathbb{Q}$.

Is possible to show (fact A in [4, Page 544]) that the inertia degree f_i is closely realted with the Frobenius morphism, in fact

$$f_i = \mathrm{ord}(F_i)$$

Theorem 3.8. *With C_β defined as before we have*

$$i_\beta = \prod_{i=1}^{s} \epsilon_i^{g_i - 1} f_i^{2g_i - 1}$$

Remark 6. This index is *optimal* because we have a lot of info about its factorization for definiton, also since ϵ_i and f_i are factors of $\phi(n)/2$, the factorization of the last one is enough to know the i_β's. We will see later that is also smaller than other index already studied.

Proof. For $s = 1$ this is trivial, since $i_\beta = 1$.

For $s \geq 2$ is possible to prove that $\epsilon_i = \phi(p_i^{e_i})$. For $i \in S$ and χ such that $p_i \nmid f_\chi$ we define

$$y(\chi, i) = \phi(p_i^{e_i}) \cdot \chi(\beta(i)) + 1 - \chi^{-1}(p_i)$$

Considering $\overline{\chi}$ to be the character induced by χ in G_i we have $\chi(\beta(i)) = \overline{\chi}(N_{F^i})$ and $\chi(p_i) = \overline{\chi}(F_i)$ using the isomorphism between G_i and the relative modulo ring. There are two cases:

$\chi(p_i) = 1$: $\overline{\chi}(N_{F_i}) = \sum \overline{\chi}(F_i)^j = \sum 1 = \mathrm{ord}(F_i) = f_i$ and so $y(\chi, i) = \phi(p_i^{e_i}) f_i + 0 = \epsilon_i f_i$

$\chi(p_i) \neq 1$: Since $\overline{\chi}(F_i) \neq 1$ follows $\overline{\chi}(N_{F_i}) = 0$. Hence $y(\chi, i) = \phi(p_i^{e_i}) \cdot \chi 0 + 1 - \chi^{-1}(p_i) = 1 - \chi^{-1}(p_i)$

Then, indexing the product by i:

$$
\begin{aligned}
i_\beta &= \prod_{i=1}^{s} \prod_{\substack{\chi \neq 1 \text{ even} \\ p_i \nmid f_\chi}} y(\chi, i) \\
&= \prod_{i=1}^{s} \left(\prod_{\chi(p_i)=1} \epsilon_i f_i \prod_{\chi(p_i)\neq 1} (1 - \chi^{-1}(p_i)) \right) \quad (\chi \neq 1 \text{ even}, \, p_i \nmid f_\chi) \\
&\overset{*}{=} \prod_{i=1}^{s} ((\epsilon_i f_i)^{g_i - 1} \cdot f_i^{g_i}) \\
&= \prod_{i=1}^{s} \epsilon_i^{g_i - 1} f_i^{2g_i - 1}
\end{aligned}
$$

In $*$ the exponent $g_i - 1$ come from 15 (there isn't the trivial character). Instead for the second part we are using that, since $\chi^{-1}(p_i) = \overline{\chi^{-1}}(F_i)$ is a non trivial $\mathrm{ord}(F_i) = f_i$-th root of unity for all characters (and varing χ we get every unit g_i times) we use that from the factorization of $x^f - 1$ as $(x-1)(1+x+...+x^{f-1})$ evaluated in 1 we have $\prod_{\zeta^f=1, \zeta \neq 1}(1 - \zeta) = f$.

□

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